J. GAMBOA*and F. MENDEZ†
Departamento de Física, Universidad de Santiago de Chile, Casilla 307, Santiago 2, Chile

The problem of time and the quantization of three dimensional gravity in the strong coupling regime is studied following path integral methods. The time is identified with the volume of spacetime. We show that the effective action describes an infinite set of massless relativistic particles moving in a curved three-dimensional target space, *i.e.* a tensionless 3-brane on a curved background. If the cosmological constant is zero the target space is flat and there is no "graviton" propagation (*i.e.*, $G[g_{ij}(2), g_{ij}(1)] = 0$). If the cosmological constant is different from zero, $3D$ gravity is both classical and quantum mechanically soluble. Indeed, we find the following results: i) The general exact solutions of the Einstein equations are singular at $t = 0$ showing the existence of a big-bang in this regime and ii) the propagation amplitude between two geometries $< g_{ij}(2), t_2 | g_{ij}(1), t_1 >$ vanishes as $t \to 0$, suggesting that big-bang is suppressed quantum mechanically. This result is also valid in $D > 3$.

I. INTRODUCTION

The quantization of the gravitational field is a problem that has resisted in spite of intense research in the last forty years [1]. The conceptual problems involved in the dynamics of gravity are main barrier towards understanding the gravitational field at the quantum level. When gravity is present, the role played by the observer is intrinsic and the concept of time is lost. This last fact implies that the evolution of the physical states, as it is generally assumed in quantum mechanics, becomes meaningless and one should try to understand the dynamics of the gravitational field in some other way. In the past, several attempts have been tried to adapt the interpretation of quantum mechanics in the presence of gravitation [2]. These attempts include incorporating the observer, or dropping the requirement of unitarity in the quantum theory [3].

The solutions to these difficulties present new and formidable problems in the quantum treatment of the fulfledged theory. This underlines the importance of studying simplified models in order to understand the key features of more complicated four dimensional problems. Along this research line, in recent years models of gravity in two and three dimensions have attracted considerable interest. Although these models do not attempt to describe the real world, they keep many of the features occurring in four dimensions, which could really be studied in the quantum version of simplified models.

Gravity in two dimensions with cosmological constant has been exactly quantized as a conformal field theory [4] and two-dimensional models describing black holes or cosmological singularities have been found [5]. Although these models still present difficulties, there is a good indication that the singularities predicted by the classical theory should be smoothed out by quantum effects.

In three dimensions similar properties have also been described. For instance, a black hole solution similar to the Kerr one has been found [6]. Also, an equivalence between $3D$ gravity and Chern-Simons theory has been noted [7] but, to our knowledge, no quantitive argument describing smoothing of singularities at the quantum level has been given so far.

This kind of argument could be useful in the understanding of the role played by the classical singularities in the quantum theory. There are two prominent examples in the history of physics about it. One of them is the stability of the hydrogen atom: the electron never falls to the nucleus because the hamiltonian is self-adjoint and the current of probability vanishes in $r = 0$. The second example is the smoothing of the electron self-energy by renomalization in quantum electrodynamics.

In order to understand conceptual issues such as time, probability and evolution of the physical states in [9] we proposed an approach to $2D$ quantum dilaton gravity where the temporal component of the ϕ^A two vector - that defines the target space - play de role of time. Then, if one assumes this point of view one can prove that indeed there is no big-bang singularity.

The purpose of the present paper is to report new results for quantum gravity in the strong coupling regime. Our main results can be summarized as follow; i) The Einstein equations in the strong coupling regime in 2+1 dimensions are solved exactly for the topology $\Re \times \Sigma$, where Σ is a Riemann surface and it is shown that there is a cosmological singularity at $t = 0$. ii) The quantum dynamics of this system is equivalent to an infinite set of relativistic massless particles moving in three dimensional spacetime (tensionless 3-brane). iii) For any D, we show that for $(\Re \times \Sigma^{D-1})$, the initial big-bang singularity is generic and smoothed out by quantum mechanics.

The paper is organized as follows. In section 2 the physical meaning of the strong coupling limit is discussed and the necessary arguments for the gauge fixing are given. In section 3, the Einstein's equations in the proper-time gauge are solved. In section 4, $3D$ gravity is

*E-mail: jgamboa@lauca.usach.cl
†E-mail: fmendez@lauca.usach.cl

quantized in the strong coupling limit using path integral methods and section 5 contains the conclusions.

II. THE STRONG COUPLING LIMIT AND THE GAUGE FIXING PROCEDURE

The action of gravity for a D-dimensional spacetime is

$$S = \int d^D x (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i), \tag{1}$$

where the constraints are defined as

$$\begin{aligned} \mathcal{H} &= G_{ijkl}\pi^{ij}\pi^{kl} + \frac{1}{G^2} g(R^{(D-1)} + 2\Lambda), \\ \mathcal{H}_i &= -2\pi^j_{i;j}, \end{aligned} \tag{2}$$

being G the Newton constant, Λ the cosmological constant and G_{ijkl} the supermetric defined as

$$G_{ijkl} = \frac{1}{2}(g_{ik}g_{jl} + g_{il}g_{jk} - \frac{2}{D-2} g_{ij}g_{kl}). \tag{3}$$

The strong coupling limit is equivalent to taking the limit $G \to \infty$ and corresponds to replace $\mathcal{H}$ in (2) by [8]

$$\mathcal{H} = G_{ijkl}\pi^{ij}\pi^{kl} + \lambda g, \tag{4}$$

where $\lambda = 2\Lambda/G^2$ being held fixed. In this limit the spatial derivatives of the metric are irrelevant and therefore the situation is equivalent to assuming that the metric of spatial sections only depends on time.

In this regime the system (4) describes an infinite set of massless relativistic particles moving in a three dimensional curved spacetime with metric G_{AB}. The additional term λg can be considered as an external potential.

The consistency of (4) in the limit $G \to \infty$, is guaranteed by the closure of the constraint algebra,

$$\begin{aligned} &[\mathcal{H}(x), \mathcal{H}(x^{'})] = 0, \\ &[\mathcal{H}(x), \mathcal{H}_i(x^{'})] = (\mathcal{H}(x) + \mathcal{H}(x^{'}))\ \delta_{,i}(x, x^{'}), \\ &[\mathcal{H}_i(x), \mathcal{H}_j(x^{'})] = \mathcal{H}_i(x)\delta_{,j}(x, x^{'}) + \mathcal{H}_j(x)\delta_{,i}(x, x^{'}). \end{aligned} \tag{5}$$

The strong coupling regime $G \gg 1$ corresponds to a very high energy region and when $G = \infty$ all the points of the spacetime become causally disconnected, or equivalently, the Lorentz group is contracted to the Carroll group [10]. Since this contraction is independent of the value of λ, (5) corresponds to the contraction of both the Lorentz and the anti-de Sitter group.

The limit $G \to \infty$ eliminates the spatial derivatives from $\mathcal{H}$. Thus, the dynamics of the gravitational field can only depend on time, while the dependence on the space coordinates is only parametric. This decoupling between space and time is particularly suited to describe a homogeneous cosmology, where the time evolution of spatial sections is a scaling transformation.

In cosmology one usually assumes isotropy and homogeneity of the spatial sections. In the proper-time gauge, *i.e.* with the Lagrange multipliers $N = g_{00}^{-1/2}$, $N_i = g_{0i}$ satisfying [11]

$$\dot{N} = 0, \qquad N_i = 0, \tag{6}$$

the space geometry takes the form

$$g_{ij} = f(t)\gamma_{ij}. \tag{7}$$

Let us consider a topology of the spacetime of the form $T \times \Sigma^{D-1}$ where Σ^{D-1} is a closed compact surface.

In order to prove that (6) is really a good gauge, we should show that (1) is invariant under the gauge transformations generated by $\mathcal{H}$ and $\mathcal{H}_i$, *i.e.*

$$\begin{aligned} \delta N &= \dot{\epsilon} - (N\epsilon^i)_{,i} + (N^i\epsilon)_{,i} \\ \delta N_i &= \dot{\epsilon}_i. \end{aligned} \tag{8}$$

The operator in (8) is always hyperbolic and can be inverted to solve for ϵ. Replacing (8) in the variation of (2) one finds

$$\begin{aligned} \delta S &= \delta\Big(\int d^D x (\pi^{ij} \dot{g}_{ij} - N\mathcal{H} - N^i \mathcal{H}_i) \Big), \\ &= \int d^{D-1}x \ \Big(\epsilon (G_{ijkl}\pi^{ij}\pi^{kl} - \lambda g) + \epsilon^i \mathcal{H}_i \Big)\Big|^{t_2}_{t_1} \\ &+ 2\int dt \int_{\Sigma_{D-1}} d^{D-1}\sigma_l (N_k \delta\pi^{kl}). \end{aligned} \tag{9}$$

In the first term, $\epsilon^i \mathcal{H}_i$ is cancelled because the diffeomorphism constraints are satisfied, while the other term is nonzero unless

$$\epsilon(t_1, \mathbf{x}) = 0 = \epsilon(t_2, \mathbf{x}). \tag{10}$$

In addition to (10), one can impose the condition

$$\epsilon^i = 0, \ \ \forall(\mathbf{x}, t), \tag{11}$$

because there is no restriction on the tangential deformations. These last requirements are consistent with the gauge conditions (6) and establish that it is a good gauge condition [12].

III. CLASSICAL COSMOLOGIES IN THE STRONG COUPLING LIMIT

In this section we look for cosmological solutions to the Einstein equations in the strong coupling limit for the topology $T \times \Sigma^{D-1}$, where Σ^{D-1} is a locally flat surface. Intuitively one expects that the removal of spatial derivatives from the $\mathcal{H}$ constraint can be formally seen as a set of infinite massless free relativistic particles moving in a D-dimensional spacetime.

Since in the strong coupling limit one neglects the spatial derivatives, a natural ansatz for the metric is

$$g_{ij} = f(t)\tilde{g}_{ij}, \tag{12}$$

where $\tilde{g}_{ij}$ is a function of $\vec{x}$ and, for $D = 3$, could be computed using the uniformization theorem [13]. However, this is not necessary because there are no spatial derivatives in $R^{(D-1)}$ and $\tilde{g}_{ij}$ only appears as a constant factor in the contraint $\mathcal{H}$ (*i.e.* only depending on $\vec{x}$).

The supermetric (3) for this geometry becomes

$$G_{ijkl} = \frac{1}{2}f^2\left(\tilde{g}_{ik}\tilde{g}_{jl} + \tilde{g}_{il}\tilde{g}_{jk} - \frac{2}{(D-2)}\tilde{g}_{ij}\tilde{g}_{kl}\right), \tag{13}$$

and the Hamiltonian constraint is

$$\begin{aligned}\mathcal{H} = &\frac{\tilde{g}^{-1}(D-2)^3(D-1)}{(4\pi N)^2}f^{(1-D)}\left(\frac{\dot{f}}{f}\right)^2 \\ &+\tilde{g}\lambda f^{(D-1)} = 0.\end{aligned} \tag{14}$$

Solving this equation one finds

$$f = \omega\ t^{\frac{1}{1-D}}, \tag{15}$$

where

$$\omega = 4\pi N\tilde{g}\sqrt{\frac{-\lambda(D-1)}{(D-2)^3}}, \tag{16}$$

Looking at (16) it is clear that the cosmological constant is necessarily negative and from (14) one see that the solution of the equation of motion contains a time singularity as in standard 3+1 cosmology.

IV. QUANTUM MECHANICS IN SUPERSPACE

The aim of this section is to write down the propagation amplitude for three- dimensional quantum gravity in the strong coupling limit which in the proper-time gauge reads

$$\begin{aligned}G[g_{ij}(2), g_{ij}(1)] = \int &\mathcal{D}N\mathcal{D}g_{ij}\mathcal{D}\pi^{ij}\delta[\dot{N}]\det(C) \\ &\times\ e^{i\int d^3x(\pi^{ij}\dot{g}_{ij} - N\mathcal{H})},\end{aligned} \tag{17}$$

where $\det(C)$ is the Faddeev-Popov determinant given by

$$\det(C) = \det\begin{pmatrix} \partial_\tau^2 & N_{,i}\,\partial_\tau + N\partial_\tau\partial_i \\ 0 & \delta_{ij}\partial_\tau\end{pmatrix}. \tag{18}$$

However this expression remains formal as long as one does not have an independent notion of time that could be used by an "external" observer. Technically speaking in ordinary quantum mechanics the external time is identified with the direction in field space which corresponds to a negative contribution to the kinetic energy in the action [1]. This fact can be easily visualized by considering the free massless relativistic particle described by the lagrangian

$$L = \frac{1}{2N}\dot{X}^\mu\dot{X}_\mu, \tag{19}$$

where N is the einbein. The temporal part of (19) is $-\frac{1}{2N}\dot{X}_0^2$ and, futhermore, one defines time as $X_0 = t$.

Our goal below will be find a similar structure for gravity.

In order to do that, let us integrate π^{ij} in (17)

$$G[g_{ij}(2), g_{ij}(1)] = \int \mathcal{D}N\mathcal{D}g_{ij}e^{i\int d^3x(\frac{1}{2N}\dot{g}_{ij}G^{ijkl}\dot{g}_{kl} - \frac{1}{2}\lambda Ng)}, \tag{20}$$

where the Faddeev-Popov determinant was absorbed as an overall normalization. The action in (20) has the structure $\dot{X}^A G_{AB}\dot{X}^B/2N$ if we identify

$$X_A \leftrightarrow g_{(ij)} \tag{21}$$

where N is the einbein and λNg plays the role of a scalar potential. Thus, (20) can be seen as describing a massless relativistic particle on a background metric G.

However, as it is well known [14], $G^{ijkl} = -\frac{1}{2}(g^{ik}g^{jl} + g^{il}g^{jk} - 2g^{ij}g^{kl})$ is the metric tensor over the three-dimensional superspace with signature $(-,+,+)$ and from here it is quite natural to interpret the volume of the space as time in quantum gravity. This interpretation is very closed related to the functional difussion equation of p-brane theory [15] where for $p = 0$ (particle) the time is the proper-time , for $p = 1$ (string) it is the area of the world- sheet, for $p = 2$ (2-brane) it is the world-volume and so on.

Thus from this point of view the loop wave equation (for zero cosmological constant) becomes

$$-G^{ijkl}\frac{\delta^2}{\delta g^{ij}\delta g^{kl}}G[g_{ij}(2), g_{ij}(1)] = i\frac{\partial}{\partial V}G[g_{ij}(2), g_{ij}(1)], \tag{22}$$

where in the RHS the Moser's theorem has been used [16].

It is interesting to observe that in the absence of a cosmological constant, three-dimensional quantum gravity in the strong coupling limit is exactly mapped to a tensionless 3-brane moving on a curved background. However as in three dimensions $\Lambda = 0$ implies $R^{(3)} = 0$, then for the topology $M(g) = \Re\times\Sigma$ the Gauss- Codazzi equations imply that $R^{(2)} = 0$, *i.e.* $g_{ij} = \delta_{ij}$.

Now, if we posit the correspondence

$$M(g) \sim M(G), \tag{23}$$

where $M(G)$ is the manifold of the target space and Σ and Σ' are flat Riemann surfaces in $M(g)$ and $M(G)$. The propagation amplitude of geometries becomes

$$G[g_{ij}(2), g_{ij}(1)] = 0, \tag{24}$$

that is, there are no gravitons propagating.

This result subsists for any genus because the strong coupling limit does not permit the formation of new geometrical structures.

If the cosmological constant is different from zero, the problem can be solved using the fact that in two dimensions the topology of the surface is classified by the genus, so that

$$G[g_{ij}(2), g_{ij}(1)] = \sum_{genus} G_{genus}[g_{ij}(2), g_{ij}(1)], \qquad (25)$$

where $G_{genus}[g_{ij}(2), g_{ij}(1)]$ is the propagation amplitude for a given genus. Each term of the sum may be computed using the uniformization theorem [13], but in the strong coupling limit this is not necessary because the the world surface is a set of infinite world-lines. Then after to use the loop-wise expansion one has

$$\begin{aligned} G_{genus}[\ g\ _{ij}(2), g_{ij}(1)] &= \int_0^\infty \mathcal{D}N(x)\ e^{-S_{cl}} \int \mathcal{D}h_{ij} e^{-S(h)}, \\ &= \int_0^\infty \mathcal{D}N(x)\ e^{-S_{cl}(g)} \det{}^{-\frac{3}{2}} \left(\frac{\delta^2 S}{\delta g_{ij}(1) \delta g^{ij}(2)} \right) \qquad (26) \end{aligned}$$

where h_{ij} are quantum fluctuations, the Pauli-de Witt-Van Vleck determinant contains all the higher corrections in $\hbar$ and a Wick rotation has been performed.

Using (15) one finds

$$S_{cl} = \int_{-\infty}^{\infty} d\tau \left(\frac{C_0}{\tau^2} + \frac{C_1}{\tau} \right), \qquad (27)$$

where C_0 and C_1 are constants that come from the spatial integration.

The first term in (27) is divergent in $\tau = 0$ and the second is a finite contribution. Furthemore

$$G_{genus}[g_{ij}(2), g_{ij}(1)] = 0, \qquad (28)$$

and quantum mechanically the big-bang singularity disappears.

One could note that for $D > 3$, (25) also vanishes implying the smoothing-out of the big-bang in higher dimensions.

V. CONCLUSIONS

In conclusion, our results suggest no initial singularity for the universe. Probably at the beginning, the universe was regular but in unstable equilibrium and evolved to the present expansion by quantum fluctuations.

As a technical final note we would like to mention that when spatial corrections are included, the calculation of (25) is more involved because the integration in g_{ij} is highly non-trivial due to the no existence of a theorem of classification of Riemann surfaces in higher dimensions. However in $D = 3$ these corrections can be taken in account [17]-by using the uniformization theorem- and similar quantum effects for spatial singularities (black- holes) should be found. We are trying to understand this issue.

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